Sunday Times Teaser 3191 – The Budgie's Extra Ration

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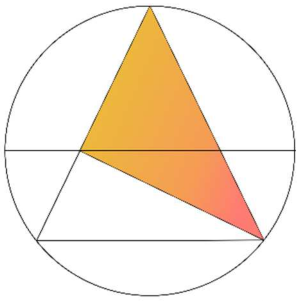
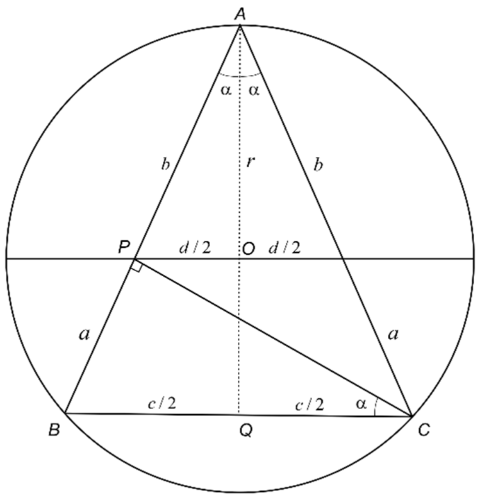


The budgie's circular toy hung on a hook. Two equal legs suspended his budgerigar-seed dispensing chord from that hook. Parallel to the chord, a diameter crossed the middle. When budgie knocked his seed dispenser below the diameter, a triangle lit up, as shown, and he got an extra seed ration. In the design of the toy, the ratio of the length of the chord to the length of the smallest side of the lit triangle has been adjusted so that a right-angled triangle results, with the right angle on the diameter

What is the square of that ratio?

Solution by Peter Noll



With notation as shown in the diagram on the right we can use the similar triangles *ABQ, APO* and *CBP* to give the relationships:

$$\frac{a}{c} = \frac{c/2}{a+b} \quad \rightarrow \quad c^2 = 2a(a+b)$$

$$\frac{d}{c} = \frac{b}{a+b} \quad \rightarrow \quad d^2 = \frac{2ab^2}{a+b}$$

$$(2r)^2 = (2b)^2 - d^2 = 2b^2\left(\frac{a+2b}{a+b}\right)$$

So c, d and r can be expressed in terms of a and $\epsilon = b/a$:

$$\left(\frac{c}{a}\right)^2 = 2(1+\epsilon) \tag{1}$$

$$\left(\frac{d}{a}\right)^2 = \frac{2\epsilon^2}{(1+\epsilon)} \tag{2}$$

$$\left(\frac{2r}{a}\right)^2 = 2\epsilon^2\ \frac{(1+2\epsilon)}{(1+\epsilon)} \tag{3}$$

Using the equation for the circumcircle of a triangle in terms of its sides and area now gives:

$$2r = \frac{(a+b)^2c}{c\sqrt{(b+a)^2-(c/2)^2}} = \frac{a(1+\epsilon)^2}{\sqrt{(1+\epsilon)^2-(1+\epsilon)/2}} \rightarrow \left(\frac{2r}{a}\right)^2 = \frac{2(1+\epsilon)^3}{(1+2\epsilon)} \tag{4}$$

Equating $(2r/a)^2$ from equations (3) and (4) now gives:

$$\frac{2(1+\epsilon)^3}{1+2\epsilon} = 2\epsilon^2\left(\frac{1+2\epsilon}{1+\epsilon}\right) \rightarrow (1+\epsilon)^4 = \epsilon^2(1+2\epsilon)^2 \rightarrow (1+\epsilon)^2 = \epsilon(1+2\epsilon) \rightarrow \ \epsilon^2-\epsilon = 1$$

This is quadratic in ϵ which has the solutions $\epsilon = (1 \pm \sqrt{5})/2$ for which only the positive sign gives a positive value showing that $\epsilon = b/a\ = (1+\sqrt{5})/2$, a value known as the golden ratio and often referred to as φ. The value we are asked to provide is:

$$\left(\frac{c}{b}\right)^2 = \left(\frac{c}{a}\right)^2 \Big/ \left(\frac{b}{a}\right)^2 = \frac{2(1+\epsilon)}{\epsilon^2} = \frac{2\left(4+2+2\sqrt{5}\right)}{\left(1+\sqrt{5}\right)^2} = \frac{12+4\sqrt{5}}{6+2\sqrt{5}} = 2$$

giving the answer as 2.