Sunday Times Teaser 3191 – The Budgie's Extra Ration

By Paul Hughes

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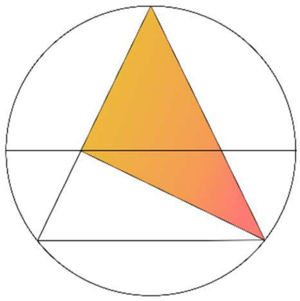
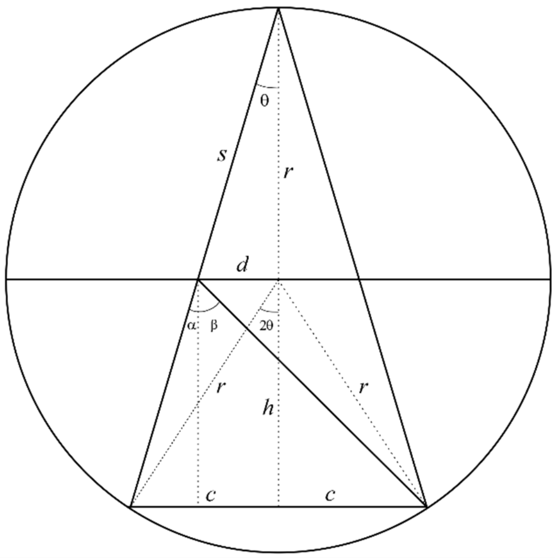


The budgie's circular toy hung on a hook. Two equal legs suspended his budgerigar-seed dispensing chord from that hook. Parallel to the chord, a diameter crossed the middle. When budgie knocked his seed dispenser below the diameter, a triangle lit up, as shown, and he got an extra seed ration. In the design of the toy, the ratio of the length of the chord to the length of the smallest side of the lit triangle has been adjusted so that a right-angled triangle results, with the right angle on the diameter

What is the square of that ratio?

Solution by Brian Gladman



With notation as shown in the diagram on the left, we are told that the angle $\alpha + \beta$ is a right angle, which means that $\cos(\alpha + \beta) = 0$. This is equivalent to:

$$\tan\alpha \tan\beta = 1$$

From the diagram:

$$\tan\alpha = \frac{c - d}{h} = \frac{\sin 2\theta - \tan\theta}{\cos 2\theta}$$

$$\tan\beta = \frac{c + d}{h} = \frac{\sin 2\theta + \tan\theta}{\cos 2\theta}$$

Combining these equations now gives:

$$\sin^2 2\theta - \tan^2\theta = \cos^2 2\theta$$

This can be simplified by rearranging and then expanding the trigonometric expressions to give:

$$\sin^2\theta = \cos^2\theta\,\{8\sin^2\theta\cos^2\theta - 1\} = (1 - \sin^2\theta)\{8\sin^2\theta - 8\sin^4\theta - 1\}$$

If we define x as $\sin^2\theta$, this can now be expressed as a polynomial in x:

$$8x^3 - 16x^2 + 8x - 1 = (2x - 1)(4x^2 - 6x + 1) = 0$$

which can be solved for x to give $x = 1/2$ and $x = \left(3 \pm \sqrt{5}\right)/4$. The first solution ($\theta = \pi/4$) does not give a meaningful configuration; the second solution with a positive sign gives $\sin\theta > 1$, which leaves $x = \left(3 - \sqrt{5}\right)/4$ as the only useful solution.

The ratio we are asked to find is:

$$\left(\frac{2c}{s}\right)^2 = \left(\frac{2r\sin 2\theta}{r/\cos\theta}\right)^2 = 16\sin^2\theta\,(1 - \sin^2\theta)^2 = 16x(1 - x)^2$$

for which substituting $x = \left(3 - \sqrt{5}\right)/4$ gives the solution to the teaser as 2. The angle θ is approximately 25.9 degrees.