Sunday Times Teaser 3191 – The Budgie's Extra Ration

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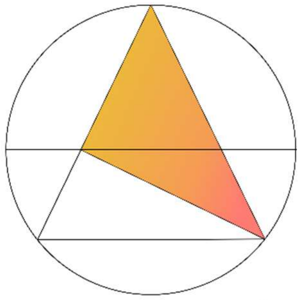
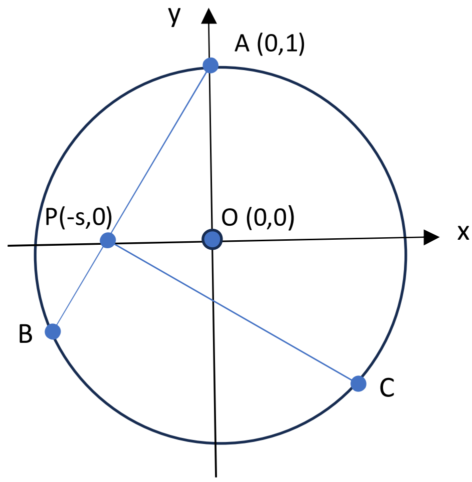


The budgie's circular toy hung on a hook. Two equal legs suspended his budgerigar-seed dispensing chord from that hook. Parallel to the chord, a diameter crossed the middle. When budgie knocked his seed dispenser below the diameter, a triangle lit up, as shown, and he got an extra seed ration. In the design of the toy, the ratio of the length of the chord to the length of the smallest side of the lit triangle has been adjusted so that a right-angled triangle results, with the right angle on the diameter

What is the square of that ratio?

Solution by Ciaran Lewis



The basic problem is depicted in the diagram showing a unit circle with a chord AB. Lines AB and PC are perpendicular and, by sliding point P(-*s*,0) along the *x*-axis, points B and C both move clockwise or anti-clockwise around the lower quadrants. The aim is to find a value for *s* such that B and C are equidistant from the *x*-axis (*i.e.* $|B_x| = C_x$ and $B_y = C_y$) with the chord BC parallel to the diameter through P. The lengths AP and CB can then be determined and the ratio R=CB/AP calculated.

The product of the gradients for lines AB and PC is -1.

Then for AB, $y = x/s + 1$ and for PC, $y = -sx - s^2$

Substituting each in turn into $x^2 + y^2 = 1$ we find the quadratic equations defining B_x and C_x.

For B_x we have $x((1+s^2)x + 2s) = 0$

Roots are then 0 or $-\left(\dfrac{2s}{1+s^2}\right)$ with $|B_x| = \left(\dfrac{2s}{1+s^2}\right)$

For C_x we have $(1+s^2)x^2 + 2s^3x - (1-s^4) = 0$

Roots are then $\dfrac{\left(-s^3 \pm \sqrt{1+s^2-s^4}\right)}{1+s^2}$ with $C_x = \dfrac{\left(-s^3 + \sqrt{1+s^2-s^4}\right)}{1+s^2}$

Equating $|B_x|$ and C_x, we have the condition $2s + s^3 = \sqrt{1+s^2-s^4}$

After squaring we have $s^6 + 5s^4 + 3s^2 - 1 = 0$ which defines the required value for *s*.

To simplify, let $S \equiv s^2$ and then we have a cubic equation to solve.

The cubic equation is $S^3 + 5S^2 + 3S - 1 = 0$ and we see, by inspection, that one solution is $S = -1$

Hence we see that the cubic equation can be expressed as $(S+1)(S^2 + aS - 1) = 0$, where *a*=4.

Roots of the quadratic factor are $S = -2 \pm \sqrt{5}$ so we see S can have two negative and one positive value. Since $S \equiv s^2$ the latter is needed; *i.e.* the teaser solution requires $S = s^2 = \sqrt{5} - 2$

Since $AP = \sqrt{1+s^2}$ and $BC = 4s/(1+s^2)$ we have $R^2 = 16s^2/(1+s^2)^3 = 16S/(1+S)^3$

Finally we have $R^2 = 16(\sqrt{5}-2)/(\sqrt{5}-1)^3 = 16(\sqrt{5}-2)/(5\sqrt{5} - 15 + 3\sqrt{5} - 1) = 2$