Sunday Times Teaser 3178 – Drying Boards

by Mark Valentine

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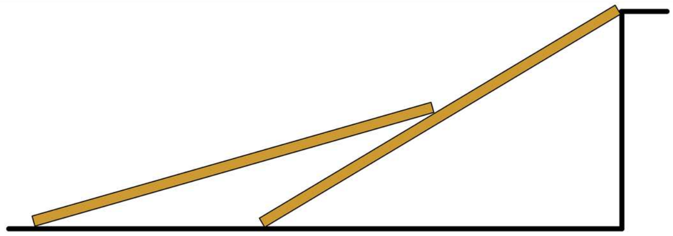
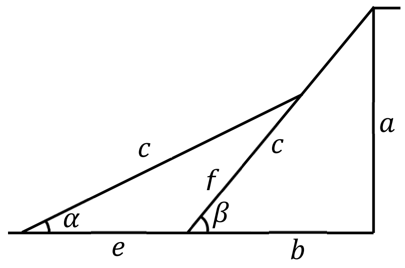


Chef Ignacio liked to prop his two identical thin rectangular chopping boards against the shelf at the end of his counter to dry. He placed the top of the first one flush with the shelf corner and rested the second on the first, as shown in the diagram. To aid drying, he positioned the second to maximise the air volume in the bounded region below it. The length of each board is an even number of cm (less than 26cm) and the height of the shelf above the counter is an integer number of millimetres.

The distance between the points at which the two boards touch the counter was a whole number of millimetres. What was this distance?

Solution by Brian Gladman



The left-hand board has to be placed in a way that maximises the area of the triangle with sides (c, e, f) and, since c and the angle β are fixed, the lengths e and f have to be adjusted to achieve this. The area (A) of this triangle is $(1/2)ef \sin\beta$ which we can differentiate and equate to zero to find the position of a minimum or maximum:

$$\partial A = (1/2)(e\,\partial f + f\,\partial e)\sin\beta = 0$$

We also have:

$$c^2 = e^2 + f^2 + 2ef\cos\beta$$

which we can differentiate to show that:1

$$(e + f\cos\beta)\partial e + (f + e\cos\beta)\partial f = 0$$

We can now eliminate the differentials:

$$-\frac{\partial e}{\partial f} = \frac{e}{f} = \frac{f + e\cos\beta}{e + f\cos\beta} \;\Rightarrow\; e^2 - f^2 = 0 \;\Rightarrow\; e = \pm f$$

to show that the maximum area occurs when e and f are equal, showing that the triangle (c, e, f) is isosceles and also that $\beta = 2\alpha$.

We can now express the cosines of α and 2α as:

$$\cos\alpha = \frac{c}{2e} \quad ; \quad \cos 2\alpha = \frac{b}{c}$$

and then use the trigonometric identity $\cos 2\alpha = 2\cos^2\alpha - 1$ to show that:

$$e = c\sqrt{\frac{c}{2(c+b)}}$$

Since the sides (a, b, c) form a right-angled triangle with integer sides a and c, this triangle is Pythagorean with (a, b, c) forming a Pythagorean triple with $c < 260$ (in millimetres). We only need to consider primitive triangles and then scale them by a multiplier (m) to give potential solutions. The fraction inside the root is in its lowest terms (since c is odd in a primitive triangle and $c + b$ cannot divide c) which means that both the numerator c and the denominator $2(c + b)$ must be perfect squares in order to make e rational.

The only candidate triples with $c < 260$ are $(7, 24, 25)$ and $(119, 120, 169)$ with root values $(5/8$ or $5/7\sqrt{2})$ and $(13/24$ or $13/17\sqrt{2})$. The rational solutions are hence $e = (125m/8)$ and $e = (2197m/24)$, but only the former with a multiplier of $m = 8$ is less than 26 cm giving the teaser solution as $e = 125$mm (the scaled triangle has dimensions $a = 192$mm, $b = 56$mm and $c = 200$mm).