Sunday Times Teaser 3165 – Round the Bend

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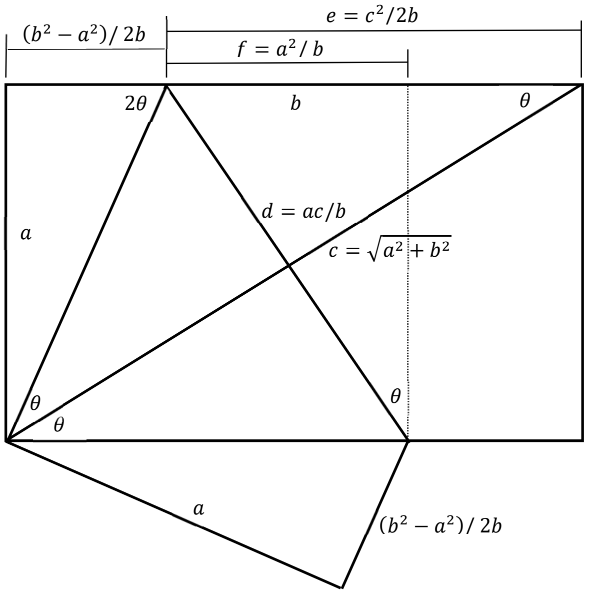
My new craft project involves card folding and requires a certain amount of precision and dexterity. For the penultimate stage a rectangular piece of card, with sides a whole number of centimetre (each less than 50cm), is carefully folded so that one corner coincides with that diagonally opposite to it. The resulting five-sided polygon also has sides of integer lengths in cm. The perimeter of the polygon is five twenty-eighths smaller than that of the perimeter of the original rectangular card.

As a final check I need to find the new area of the card.

What, in square centimetres, is the area of the polygon?

Solution by Brian Gladman



The dimensions of the rectangle will be denoted by a and b, with diagonal c and the fold line d, perpendicular to the diagonal at its midpoint. The dimensions shown on the left can all be derived using similar triangles as follows:

$$\tan\theta = a/b = f/a = (d/2)/(c/2)$$
$$\cos\theta = b/c = (c/2)/e$$

The perimeter lengths of the rectangle and the folded five-sided polygon can now be determined as:

$$\begin{aligned} P_{rec} &= 2(a+b) \\ P_{poly} &= 2a + (b^2 - a^2)/b + ac/b \\ &= \{(a+b)^2 + a(c-2a)\}/b \end{aligned}$$

The ratio of the two perimeters, which is $23/28$, hence provides the equation:

$$\frac{(a+b)^2 + a(c-2a)}{2(a+b)b} = \frac{23}{28}$$

which can be rearranged as:

$$14a^2 - 5ab + 9b^2 = 14ac = 14a\sqrt{a^2+b^2}$$

Squaring both sides and simplifying now gives the quartic equation:

$$81b^4 - 90ab^3 + 81a^2b^2 - 140a^3b = 0$$

which factors as:

$$b(3b - 4a)(35a^2 + 6ab + 27b^2) = 0$$

The quadratic component of this polynomial has complex roots (since it has a negative discriminant) so the only viable solution is given by $a = (3/4)b$. This gives the length of the folded polygon's smallest side as $(7/32)b$, which means that b is equal to 32 (since all sides have integer lengths less than 50).

The area of the polygon can be seen to be one half of the area of the rectangle plus the area of the upper left-hand triangle giving:

$$\Delta = \frac{ab}{2} + \frac{a(b^2 - a^2)}{4b} = \frac{a(3b^2 - a^2)}{4b} = 117\left(\frac{b}{16}\right)^2$$

which gives the solution as $468cm^2$.