Sunday Times Teaser 3157 – End-of-Season Analysis

by Howard Williams

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In our local football league each team plays each other once every season. The winner of a game is awarded one point, the loser no points and each team half a point if the game is drawn.

In the end-of-season analysis, it was found that for each team, exactly half of the points that they were awarded had been awarded for their games against the 15 teams with the lowest number of points.

How many teams were in the league?

Solution by Brian Gladman (with contributions from John Crabtree and Tony Smith).

Let the league be divided into T top teams and B bottom teams, $T + B$ teams in total.

Note that the scoring system is such that each match always provides one point that is shared between the two teams involved. As a result, when the top T teams play each other once, the number of points these teams gain as a whole is just the number of matches played, that is, $T(T-1)/2$ points. Similarly, the bottom B teams earn $B(B-1)/2$ points from their joint matches.

Since each top team plays each bottom team once, there are a total of TB points earned in all the matches played between the top T and bottom B teams. We cannot determine what proportion of these points that are earned by the top T and the bottom B teams but we do know that these teams earned half their points when they played against the bottom B teams (and hence also half their points in playing against the top T teams). Hence:

$$\frac{T(T-1)}{2} + \frac{B(B-1)}{2} = TB \tag{1}$$

which becomes:

$$(T-B)^2 = (T+B) \tag{2}$$

giving a quadratic for the difference between T and B in terms of B as:

$$(T-B)^2 - (T-B) - 2B = 0 \tag{3}$$

with the solutions:

$$T = B + \left\{ \frac{1 \pm \sqrt{8B+1}}{2} \right\} \tag{4}$$

Here, the square root term is an integer whenever B is a triangular number $b(b+1)/2$ and this results in the two possible values for T as:

$$\begin{array}{rcccl} T & = & B-b & = & b(b-1)/2 \\ T & = & B+b+1 & = & (b+1)(b+2)/2 \end{array} \tag{5}$$

This means that B is a triangular number and T is either of the two adjacent triangular numbers.

But, since the top and bottom teams have average scores of $T-1$ and $B-1$ respectively (half from playing among themselves and half from playing the other half of the league), we know that $T > B$ so we need the higher value of T.

Hence with $B = 15$ and hence $b = 5$, T is then 21 and the number of teams in the league is 36. I am content to leave it to others to show that a set of matches between 36 teams of this form is possible.