Sunday Times Teaser 3153 – Pole Position

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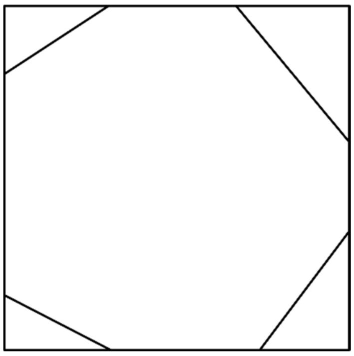


Jeb's 25x25km square ranch had his house's flagpole at the ranch's pole of inaccessibility (the point whence the shortest distance to a boundary fence was maximised).

At 50, Jeb gave each of his four sons a triangular tract of his land, with corners whole numbers of km along boundary fences from each corner post (as illustrated, not to scale). Each tract's area (in square km) equalled that son's age last birthday (all over 19, but under 30). All the tracts' perimeters differed, and each son set his hacienda's flagpole at his own tract's pole of inaccessibility.

Curiously, for Jeb's new octagonal ranch the pole of inaccessibility and the shortest distance from this to a boundary fence were unchanged.

Give the eldest son's shortest distance from his flagpole to a boundary fence.

Solution by Brian Gladman

Consider one of the son's ranches located at the south west corner of the father's ranch with the west and south edges of length a and b respectively. With the origin of a grid $(0,0)$ at the south west corner and coordinates x and y running east and north respectively, the equation of the 'hypotenuse' side of the son's ranch is then given by

$$ax + by - ab = 0 \tag{1}$$

For the hypotenuse side of the son's ranch to be further from the centre of the father's ranch than any other side, the minimum distance from the son's side to the father's centre must be larger than 12.5 kilometres. With the centre of the father's ranch at point (p,p), the equation of the minimum distance line between this point and the son's hypotenuse is the perpendicular line:

$$(y - p) = (b/a)(x - p) \tag{2}$$

Where these two lines meet x and y have the same value so we can find x at this point by eliminating y in equations (1) and (2) to give:

$$(a^2 + b^2)x = a\{ab - (a - b)p\} \tag{3}$$

With $c = \sqrt{a^2 + b^2}$ equations (2) and (3) can now be recast as:

$$c^2(p - x) = a\{(a + b)p - ab\} \tag{4}$$

$$c^2(p - y) = b\{(a + b)p - ab] \tag{5}$$

where the right-hand side values are positive since $x < p$ and $y < p$. With the minimum distance between the point (p,p) and the hypotenuse of the son's ranch as d, these two equations show that:

$$c^4\{(x - p)^2 + (y - p)^2\} = c^4d^2 = c^2\{(a + b)p - ab\}^2 \tag{6}$$

which simplifies to:

$$d = \pm\{(a + b)p - ab\}/c \tag{7}$$

To avoid the distance between the centre of the father's ranch and its sides being reduced by the son's ranches we hence need:

$$d = \{(a + b)h/2 - ab\}/c \geq h/2 \tag{8}$$

where h is the length of the sides of the father's ranch. This can be rearranged as $a + b + c \leq h$, which provides a simple criterion for the validity of the son's ranches.

When a circle centred on the point (p, p) touches the edges of the father's ranch *AND* the hypotenuse side of a son's ranch, d and p are then equal and this allows equation (7) can be solved for the resulting value of d and p. This results in two values which will be denoted by r and R:

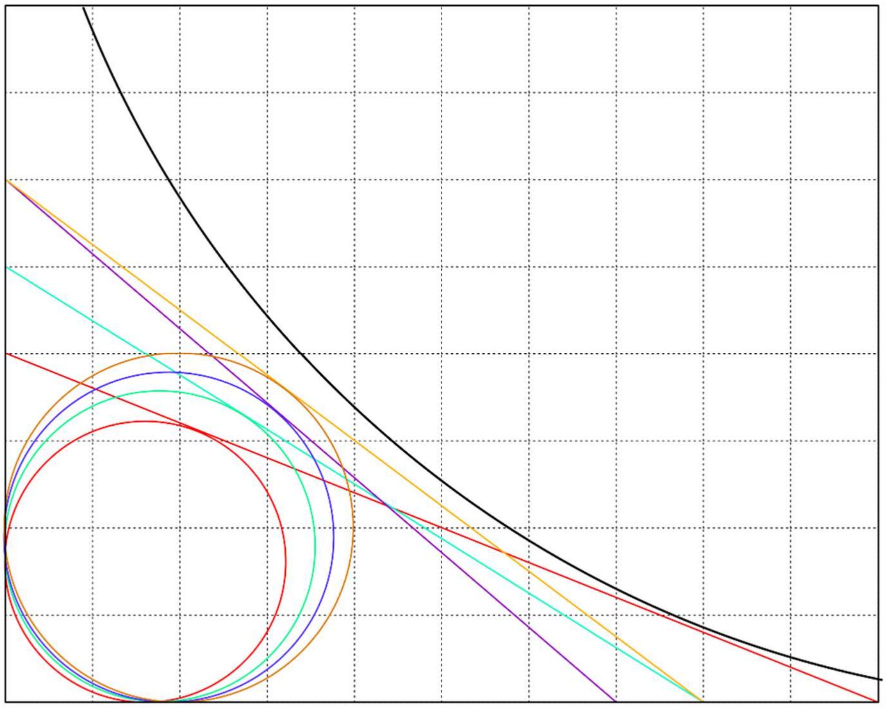
$$r_{-}, R_{+} = \frac{ab}{(a + b \pm c)} = \frac{(a + b \mp c)}{2} \tag{9}$$

This is because there are two points where a circle can touch the sides of both the father's initial and a son's eventual ranch: r is the radius of the circle within a son's ranch that touches its three sides while R is the radius of the circle within the father's ranch that touches its sides and a son's hypotenuse side[1]. Note also that $rR = ab/2$.

We now have the tools needed to solve the teaser: (a) the triangle area constraint $20 \leq area \leq 29$; and (b) the constraint $a + b + c < h$ on the perimeter of the son's ranches. Both a and b are integers and we can assume that $a < b$ which gives nine possible solutions:

a	b	Area	$c = \sqrt{a^2 + b^2}$	$a + b + c \leq 25$	r	R
4	10	20	10.770	24.770	1.615	12.385
4	11	22	11.704	26.704	1.647	13.352
4	12	24	12.649	28.649	1.675	14.325
5	8	20	9.434	22.434	1.783	11.217
5	10	25	11.180	26.180	1.909	13.090
6	7	21	9.220	22.220	1.890	11.109
6	8	24	10.000	24.000	2.000	12.000
6	9	27	10.817	25.816	2.091	12.908
7	8	28	10.630	25.630	2.185	12.815

There are five invalid solutions (in red) and four valid ones with the eldest son's ranch (in blue) having a point of inaccessibility which is 2 kilometres from its boundaries. The four valid solutions are illustrated below.



[1] In mathematical terms, the circle within the son's triangular ranch is known as the triangle's ***incircle*** while the larger circle is one of its three ***excircles***.