Sunday Times Teaser 3150 – Pyramids of Wimbledon

***by** Mark Valentine*

Published Sunday February 05 2023

Edward, the sports shop owner, had an annual display of tennis balls. He arranged the balls in four identical pyramids, with a square base holding each in place (one ball on the top of each pyramid, four on the layer below, nine below that and so on).

However, this year he wanted to display the same number of balls but reduce the total footprint of the bases by at least 55 per cent, to allow for other stock. His son Fred suggested arranging all the balls in one large pyramid with an equilateral triangular base (one ball on the top, three on the layer below, six below that and so on). Edward realised that this would work, but if there were any fewer balls, it wouldn't work.

How many balls did Edward display?

Solution by Brian Gladman

Let the number of layers of tennis balls in the square and triangular base pyramids be s and t respectively (s and t will also be used as subscripts to identify the two arrangements). The number of tennis balls in the two arrangements are given by:

$$n_s = 4\frac{s(s+1)(2s+1)}{6}$$

$$n_t = \frac{t(t+1)(t+2)}{6}$$

Taking the difference between these two expressions shows that the resulting polynomial has a factor:

$$\begin{aligned}6(n_s - n_t) &= \{4s(s+1)(2s+1) - t(t+1)(t+2)\}\\ &= (2s-t)(4s^2+6s+t^2+3t+2st+2)\end{aligned}$$

which shows that the two arrangements have an equal number of tennis balls when $\mathrm{t} = 2\mathrm{s}$. There are no further positive roots since the other factor has no negative terms.

With the radius of a tennis ball as r, the areas of the bases of the two arrangements are:

$$A_s = 4(2sr)^2 = 16s^2r^2$$

$$A_t = (\sqrt{3}/4)\{2(\mathrm{t}-1+\sqrt{3})\mathrm{r}\}^2$$

Now, with A_t/A_s as ρ, and noting $\mathrm{t} = 2\mathrm{s}$ and $\rho < 0.45$, we can use these expressions to show that:

$$\frac{2s-1+\sqrt{3}}{4s} = \pm\sqrt{\rho/\sqrt{3}}$$

which can be rearranged to provide a value for s (⌈x⌉ is the integer ≥ x)):

$$\mathrm{s} = \left\lceil \frac{\sqrt{3}-1}{2\left(2\sqrt{\frac{\rho}{\sqrt{3}}}-1\right)} \right\rceil$$

where the sign of the square root has been chosen to produce a positive result.

Setting $\rho = 0.45$ in this equation gives $\mathrm{s} = 19, \mathrm{t} = 38$ and hence the number of tennis balls displayed as 9880.