Sunday Times Teaser 3148 – Quiz Probabilities

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Each of four contending couples in a quiz game has equal probability of elimination at the end of each of the first three rounds, one couple going after each round. In the fourth round, the remaining couple has a constant probability p, less than ½, of winning the jackpot, which consists of £1000 in the first game; if the jackpot is not won, it is added to the £1000 donated in the next game. Each couple may enter three successive games of the quiz, except that any couple having played for the jackpot in the fourth round of any game then withdraws altogether, being replaced by a new couple in the next game.

If the probability that a couple, competing from the first game, wins £2000 is 7/96, what is the value of p as a fraction?

Solution by John Crabtree and Brian Gladman

All couples are equal so we can focus on one of them in calculating the outcome of the games. Each couple has a $1/4$ probability of reaching the last round of a game and hence dropping out with a win or a loss. Hence the probability of a couple playing in the next game is $3/4$. When doing so, the potential winnings depend on the way the previous game ends since a win in the previous game empties the jackpot (probability p) whereas a loss causes a rollover with the prize being added to the jackpot for the next round (probability $1-p$).

The probabilities of winning various amounts (in base prize units)

To win nothing, a couple have to get to the final and lose; continue to the second game, get to the final and lose again; continue to the third game, get to the final and lose. Hence:

$$P_0[p]=\left(\frac{1}{4}\right)(1-p)+\left(\frac{3}{4}\right)\left\{\left(\frac{1}{4}\right)(1-p)+\left(\frac{3}{4}\right)\left\{\left(\frac{1}{4}\right)(1-p)+\left(\frac{3}{4}\right)\right\}\right\}=1-\left(\frac{37}{64}\right)p$$

To win 1 prize unit, a couple have to win the first game; or win the second game after the previous game was won; or win the third game after the previous game was won. Hence:

$$P_1[p]=\left(\frac{1}{4}\right)p+\left(\frac{3}{4}\right)\left\{p\left(\frac{1}{4}\right)p\right\}+\left(\frac{3}{4}\right)^2\left\{p\left(\frac{1}{4}\right)p\right\}=\frac{21p^2+16p}{64}$$

To win two prize units, a couple have to get through a loss by other couples and then win; or get through a win and then a loss by other couples and then win. Hence:

$$P_2[p]=\left(\frac{3}{4}\right)(1-p)\left(\frac{1}{4}\right)p+\left(\frac{3}{4}\right)p\left(\frac{3}{4}\right)(1-p)\left(\frac{1}{4}\right)p=\frac{-9p^3-3p^2+12p}{64}$$

To win three prize units, a couple have to get through two losses by other couples and then win. Hence:

$$P_3[p]=\left(\frac{3}{4}\right)^2(1-p)^2\left(\frac{1}{4}\right)p=\frac{9p^3-18p^2+9p}{64}$$

We can check these probabilities by adding them to get a sum of 1.

It will also be useful to list the probabilities for winning “one or more” and “two or more” prize units:

$$P_{1+}[p]=\frac{37p}{64}$$

$$P_{2+}[p]=\frac{-21p^2+21p}{64}$$

Analysing the Teaser

If the probability of winning £2000 (2 prize units) is $q/96$ then we have the polynomial equation:

$$\frac{-9p^3 - 3p^2 + 12p}{64} = \frac{q}{96}$$

which simplifies to:

$$27p^3 + 9p^2 - 36p + 2q = 0$$

For the teaser q is 7, for which this polynomial has one negative real root and two complex roots, none of which can solve the teaser. In practice it seems likely that the probability for p set by the author will be a rational fraction so it makes sense to consider other q values to see if any offer solutions of this form. When this is done, two possible solutions emerge: $p = 1/3$ when $q = 5$ and $p = 2/3$ when $q = 6$. The teaser hence has a solution with $p = 1/3$ when the $7/96$ specified in the teaser is replaced by $5/96$.

It hence seems that the teaser author has made an error in creating this teaser.

An alternative interpretation of the teaser text is to consider winning £2000 to mean "at least £2000" which gives the equation:

$$\frac{-21p^2 + 21p}{64} = \left(\frac{7}{96}\right)$$

which simplifies to:

$$p^2 - p = -2/9$$

Multiplying this by 4 and adding 1 to each side now gives:

$$4p^2 - 4p + 1 = (2p - 1)^2 = (1/3)^2$$

and hence $p = (1 \pm 1/3)/2$ giving solutions of $p = 1/3$ and $p = 2/3$, the first of which satisfies the teaser constraints.

From these results we can see that there are two options for rescuing the teaser. It would be nice to see a cubic involved in place of the not infrequent appearance of quadratics in teasers.