Sunday Times Teaser 3147 – Noteworthy

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Apparently in Costa Lotta a single-digit percentage of banknotes are forgeries and so I have designed a marker pen which tests whether notes are genuine. I thought it would be quite useful to the banks because, on average, for every N uses it only gives an incorrect result once (where N is some whole number).

Unfortunately my design has been abandoned by the banks because it turns out that on average for every N occasions on which the pen indicates a forgery, only one of the notes will in fact be forged!

What is N?

Solution by Brian Gladman

Let $\boldsymbol{d}$ be the single digit forgery percentage and $\boldsymbol{b}$ (bad) be its fraction form ($d/100$). Let $\boldsymbol{f_p}$ and $\boldsymbol{f_n}$ be the fraction of forged notes wrongly classified as genuine and the fraction of genuine notes wrongly classified as forgeries respectively. Let the number of tests conducted be $\boldsymbol{t}$. Hence:

(1) The number of genuine notes wrongly tested as forgeries: $\boldsymbol{f_n(1-b)t}$

(2) The number of genuine notes correctly tested as genuine: $\boldsymbol{(1-f_n)(1-b)t}$

(3) The number of forgeries wrongly tested as genuine: $\boldsymbol{f_p bt}$

(4) The number of forgeries correctly tested as forgeries: $\boldsymbol{(1-f_p)bt}$

The test error rate [(1) + (3)] is $\boldsymbol{1/N}$:

$$\boldsymbol{f_n(1-b)t + f_p bt = t/N \;\Rightarrow\; f_n(1-b) + f_p b = 1/N}$$

In $\boldsymbol{N}$ tests indicating forgeries [(1) + (4)] only one test was correct:

$$\boldsymbol{f_n(1-b)t = N-1 \;;\; (1-f_p)bt = 1 \;\Rightarrow\; f_n(1-b) = (N-1)(1-f_p)b}$$

If we assume now that the two error rates ae equal ($\boldsymbol{f_p = f_n = f}$) the equations then become:

$$\boldsymbol{f = 1/N \;;\quad (1-b)f = (N-1)(1-f)b}$$

These simplify to give:

$$\boldsymbol{N = \sqrt{(1-b)/b} + 1 = \sqrt{100/d - 1} + 1}$$

which provides the solution $\boldsymbol{N = 8}$ **for** $\boldsymbol{d = 2\%}$.

Different False Positives and Negatives

Eliminating $\boldsymbol{f_n}$:

$$\mathbf{1}/\boldsymbol{N} - \boldsymbol{f_p b} = (\boldsymbol{N} - \mathbf{1})(\mathbf{1} - \boldsymbol{f_p})\boldsymbol{b} \;\; \Rightarrow \;\; \mathbf{1} - \boldsymbol{N f_p b} = \boldsymbol{N}(\boldsymbol{N} - \mathbf{1})(\mathbf{1} - \boldsymbol{f_p})\boldsymbol{b}$$

Collecting terms and simplifying:

$$\boldsymbol{N}(\boldsymbol{N} - \mathbf{1}) - \mathbf{1}/\boldsymbol{b} = \boldsymbol{N}(\boldsymbol{N} - \mathbf{2})\boldsymbol{f_p}$$

In terms of forgery percentage:

$$(\mathbf{1} - \boldsymbol{f_p})\boldsymbol{N}^{\mathbf{2}} + (\mathbf{2}\boldsymbol{f_p} - \mathbf{1})\boldsymbol{N} - \mathbf{100}/\boldsymbol{d} = \mathbf{0}$$

$$\boldsymbol{f_p} = \frac{\boldsymbol{N}^{\mathbf{2}} - \boldsymbol{N} - \mathbf{100}/\boldsymbol{d}}{\boldsymbol{N}(\boldsymbol{N} - \mathbf{2})}$$

This has a solution for $\boldsymbol{f_p} = \mathbf{0}$:

$$\boldsymbol{N} = \frac{\sqrt{\mathbf{400}/\boldsymbol{d} + \mathbf{1}} + \mathbf{1}}{\mathbf{2}}$$

which gives the result $\boldsymbol{N} = \mathbf{5}$ for a forgery percentage $\boldsymbol{d} = \mathbf{5\%}$. It is not, however, the intended solution.

Eliminating $\boldsymbol{f_p}$ instead:

$$\boldsymbol{f_n}(\mathbf{1} - \boldsymbol{b}) = (\boldsymbol{N} - \mathbf{1}) - (\boldsymbol{N} - \mathbf{1})\left(\frac{\mathbf{1}}{\boldsymbol{N}} - \boldsymbol{f_n}(\mathbf{1} - \boldsymbol{b})\right)$$

gives another result:

$$\boldsymbol{N} = \frac{\mathbf{1}}{\sqrt{\frac{\mathbf{1}}{\boldsymbol{f_n}(\mathbf{1} - \boldsymbol{b})} + \mathbf{1}}} + \mathbf{1} = \sqrt{\frac{\mathbf{100} - \boldsymbol{d}}{\mathbf{100} - \boldsymbol{d} + \mathbf{100}/\boldsymbol{f_n}}} + \mathbf{1}$$

but in this case there is no solution other than $\boldsymbol{N} = \mathbf{1}$ when $\boldsymbol{f_n} = \mathbf{0}$.