Sunday Times Teaser 3146 – Curling League

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In our curling league (with at least 5 and at most 25 teams), each team plays each other once. Teams are ranked according to the number of wins (draws are impossible). If any teams are tied on wins, ranking is only possible if those teams have different numbers of wins in their mutual games. For example, in a three-way tie if A beats B, B beats C and A beats C, the ranking is ABC, but if C beats A (or A has not yet played C), then ranking is impossible, as A and B have one win each.

At one point (each team had played G games), ranking the teams as above was possible. However, if each team had played G-1 games, a ranking would have been impossible, irrespective of results. With one more team in the league, the minimum number of games needed to allow a ranking is G+2.

How many teams are in the league and what was the value of G?

Solution by John Crabtree

An odd number of teams cannot all play the same odd number of games. Hence, since we are considering both G and $G-1$ games for each team, the number of teams (T) is even.

An odd number of teams can only all play an even number of games. Since we are considering leagues with T and $T+1$ teams, G is hence even.

The key to this teaser is to work out the maximum number of teams that can all be ranked for any even G. For teams with equal numbers of wins, we need to determine how many teams can be tied whilst still being capable of being ranked. A ranking will only be possible in these tied groups if all the teams in the group play each other, the ranking being dependent on the outcomes of these mutual games. For tied teams with less than $G/2$ wins, the size of the group is limited by the number of wins. For tied teams with $G/2$ wins, one team within the group must win $G/2$ games and another lose must lose $G/2$ games. When the number of wins is greater than G/2, the size of the group is limited by the number of losses.

For example, consider $G=8$. The maximum number of teams for each tied number of wins is:

Wins	Condition for being able to rank games between teams in the tied group
0	only one team can have no wins
1, 1	one team wins one group game (and hence the other loses one)
2, 2, 2	one team wins two group games, another wins one and a third loses both
3, 3, 3, 3	three teams win 3, 2 and 1 group games respectively, one loses all of them
4, 4, 4, 4, 4	four teams win 4, 3, 2 and 1 of the group games respectively, one loses all of them
5, 5, 5, 5	three teams lose 3, 2 and 1 of the group games respectively, one wins all of them
6, 6, 6	one team loses two group games, another loses one and a third wins both
7, 7	one team loses one game in the group games (and hence the other wins one)
8	only one team can have eight wins

By inspection the average wins per team is G/2, which is as it should be. The maximum number of teams is hence given by:

$$T_{max} = 2\sum_{i=1}^{G/2} i + \left(\frac{G}{2}+1\right) = (G+2)^2/4$$

We are looking a value of T which is an even perfect square in the range $5 \le T \le 25$. Hence $G = 2$ mod 4 giving the solution as a league with 16 teams, each playing 6 games (i.e. $G=6$).

For 17 teams, the minimum number of games to get a ranking must be even, i.e. 6 + 2 = 8, which is as required. As an aside it can be shown that for G odd the maximum even number of teams is given by $T_{max} = (G+1)(G+3)/4$.