Sunday Times Teaser 3143 – Pipe Fittings

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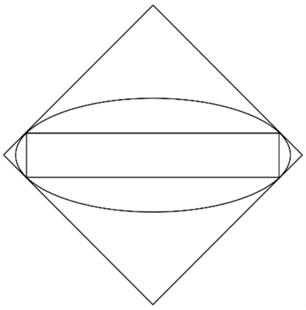
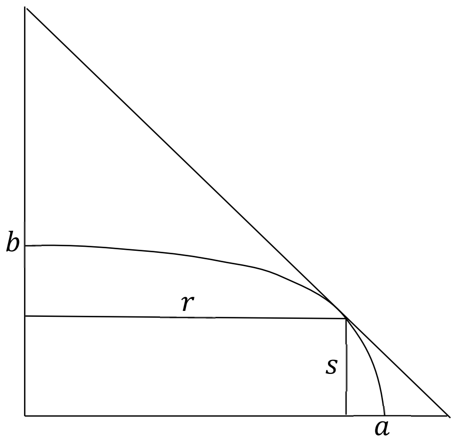


A plumber had three thin metal pipes with square, rectangular and elliptical cross-sections. In order to fit them into his van, he slid the rectangular pipe inside the elliptical pipe and the elliptical pipe inside the square pipe, before placing the pipe assembly in the van. There are four points where the pipes all touch, as shown in the diagram. The maximum and minimum widths of the elliptical and rectangular pipes and the diagonal width of the square pipe were all even numbers of mm less than 1,000, of which one was a perfect square.

What were the five widths (in increasing order)?

Solution by Brian Gladman



If the centre of the diagram is placed at the origin $(0,0)$ of an (x, y) graph, we can consider its upper right quadrant as shown on the left. The meeting point of the three pipes is at coordinates (r, s) and the semi-major and semi-minor axes of the ellipse are a and b respectively. We can also see that the semi-diagonal of the square is $r + s$. The ellipse is represented by the function:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The slope of the ellipse and the square side at the touching point are the same and we can find the slope of the ellipse by differentiating the above equation:

$$\left(\frac{2x}{a^2}\right) dx + \left(\frac{2y}{b^2}\right) dy = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{xb^2}{ya^2}$$

Equating the two slopes at the point (r, s) now gives $rb^2 = sa^2$ which can be used with the equation of the ellipse at (r, s) to eliminate s to show that:

$$r = \frac{a^2}{\sqrt{a^2 + b^2}}$$

The equation for s follows in the same way:

$$s = \frac{b^2}{\sqrt{a^2 + b^2}}$$

The semi-diagonal of the square is $r + s$ which evaluates to $\sqrt{a^2 + b^2}$ which we will define as c so that we have the Pythagorean triple (a, b, c) where the point (r, s) is given by $(a^2/c, b^2/c)$.

The five lengths we need to solve the teaser are:

$$(2a, 2b, 2\,a^2/c, 2b^2/c, 2c)$$

If we consider only primitive Pythagorean triples, a and b can't be divisible by c so we have to scale the values to obtain integer lengths:

$$(2ac, 2bc, 2a^2, 2b^2, 2c^2)$$

These now have to be scaled further until the result has only a single perfect square. Taking the first primitive triple $(3,4,5)$ gives the sorted values $(18, 30, 32, 40, 50)$ where the first scaling that gives a single perfect square is the desired solution:

$$(180, 300, 320, 400, 500)$$

The next primitive triple $(5,12,13)$ does not yield a solution since it offers no perfect squares and a maximum value of 338. No higher primitive triples offer solutions since their maximum values are all above 500.