Sunday Times Teaser 3129 - Bounce Count

by Mark Valentine

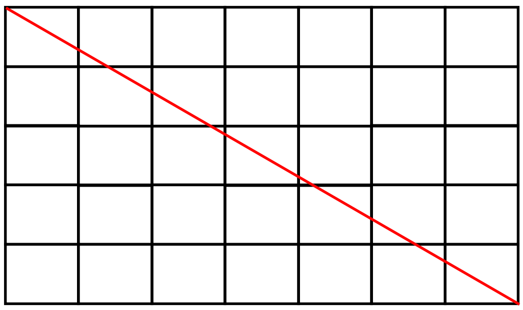
Published Sunday September 11 2022

At the local arcade, Claire and David played an air hockey game, consisting of a square table with small pockets at each corner, on which a very small puck can travel 1m left-right and 1m up-down between the perimeter walls. Projecting the puck from a corner, players earn a token for each bounce off a wall, until the puck drops into a pocket.

In their game, one puck travelled 1m farther overall than its left-right distance (for the other, the extra travel was 2m). Claire's three-digit number of tokens was a cube, larger than David's number which was triangular (1+2+3+...). Picking up a spare token, they could then arrange all their tokens into a cube and a square combined.

How many tokens did they end up with?

Solution by Brian Gladman



The puck bouncing between the walls of the square table can be represented by treating the walls as mirrors in which the puck is 'reflected' off the walls. Then, instead of reflecting the puck, we can 'reflect' the table to create an infinite grid of table reflections in which the puck travels in a straight line as shown on the left. Here horizontal and vertical lines represent bounces while the horizontal and vertical dimensions give the pucks total distances travelled left/right and up/down. The red line gives the pucks total distance travelled. Note that distances of x (left/right) and y (up/down) are integers and correspond to puck bounces of $x-1$ and $y-1$ respectively. Note also that if x and y have a common factor greater than 1, the puck will drop off the table prematurely. Let the puck travel a total distance z, an integer distance d further than its left/right travel, so that z is integer and:

$$(z-d)^2+y^2=z^2 \tag{1}$$

which gives:

$$z=(y^2+d^2)/2d \tag{2}$$

We can now determine the corresponding token value T as:

$$T=(z-d-1)+(y-1)=(y+d)^2/2d-(d+2) \tag{3}$$

Setting the d values as 1 and 2, we hence obtain:

$$T \;=\; \begin{cases} (y+1)^2/2-3 & \text{if } d=1 \\ (y+2)^2/4-4 & \text{if } d=2 \end{cases} \tag{4}$$

Claire has a three digit T value that is a cube. Testing the integer cubes between 100 and 1000 quickly finds a single solution when $d=1$ and $y=15$ with $T=125$ and $x=112$. The puck stays on the table since x and y are co-prime.

For David's values with $d=2$, testing triangular numbers below 125 gives two possibilities: $y=8$ with $T=21$ and $x=15$ or $y=12$ with $T=45$ and $x=35$. Again, the puck remains on the table.

We now need to find a combined score which, with a spare token, is the sum of a cube and a square. Hence, we have two possible solutions $125+21+1=147$ or $125+45+1=171$. Considering the cubes of 1 to 5 quickly reveals only one result as $171=3^3+12^2$ giving the solution as 125 for Claire and 45 for David.