Sunday Times Teaser 3121 – Top Marks

By Howard Williams

Published Sunday July 17 2022

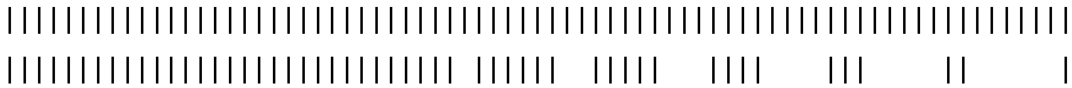
A teacher is preparing her end of term class test. After the test she will arrive at each pupil's score by giving a fixed number of marks for a correct answer, no marks if a question is not attempted, and deducting a mark for each incorrect answer. The computer program she uses to prepare parents' reports can only accept tests with the number of possible test scores (including negative scores) equal to 100.

She has worked out all possible combinations of the number of questions asked and marks awarded for a correct answer that satisfy this requirement, and has chosen the one that allows the highest possible score for a pupil.

What is that highest possible score?

Solution by Brian Gladman

Let the number of questions be n and the number of marks for a correct answer be m, with the possible marks for the test hence being between $-n$ and $m.n$ inclusive. If all questions are answered correctly the resulting mark will be $m.n$; if all but one are answered correctly, leaving one wrong or unanswered, there are two possible marks $(n-1)m$ or $(n-1)m-1$. For all but two correct there are three resulting marks. To illustrate this situation, we can plot the marks on the number line from $-n$ to $n.m$ as follows for the example $m = 7$ and $n = 9$:



We can now see that there are gaps in the full range of numbers that are not possible, the lengths of the gaps being $1, 2, \ldots, m-1$ with the total number missing being $m(m-1)/2$ (this applies provided that $n \geq m$).

We hence find that the number of possible different resulting marks (say M) is given by:

$$M = n + 1 + m.n - m(m-1)/2$$

We can now re-arrange this equation to solve for n in terms of m and M:

$$n = \frac{2M + m(m-1) - 2}{2(m+1)}$$

where we want to find a value of m for which $m.n$ has the maximum value. Checking for $M = 100$ with increasing m values yields integer results for (m, n, max) = (4, 21,84) and (7, 15,105) leading to the maximum marks for the test as 105.

Here is an alternative final derivation from John Crabtree using a re-arrangement of the above formula:

$$2n + 3 = \frac{2M}{(m+1)} + (m+1)$$

where either $2M/(m+1)$ or $(m+1)$ must be odd. Hence the pair $[(m+1), 2M/(m+1)]$ is one of (1, 200), (5,40) or (25, 8) with $(m, n, max) = (0, 99, 0), (4, 21, 84)$ or $(7, 15, 105)$, leading to 105 as the answer.