Sunday Times Teaser 3123 - A Six-Pipe Problem

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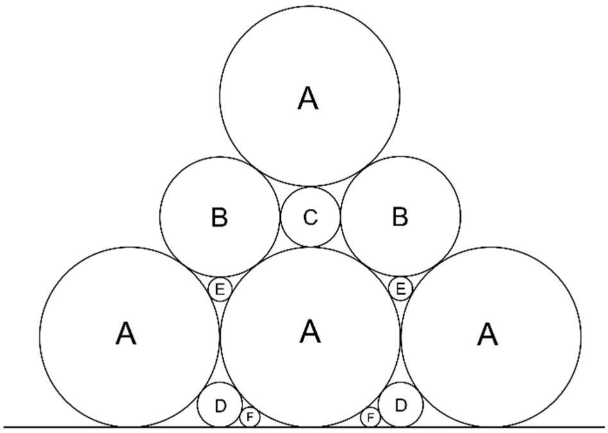
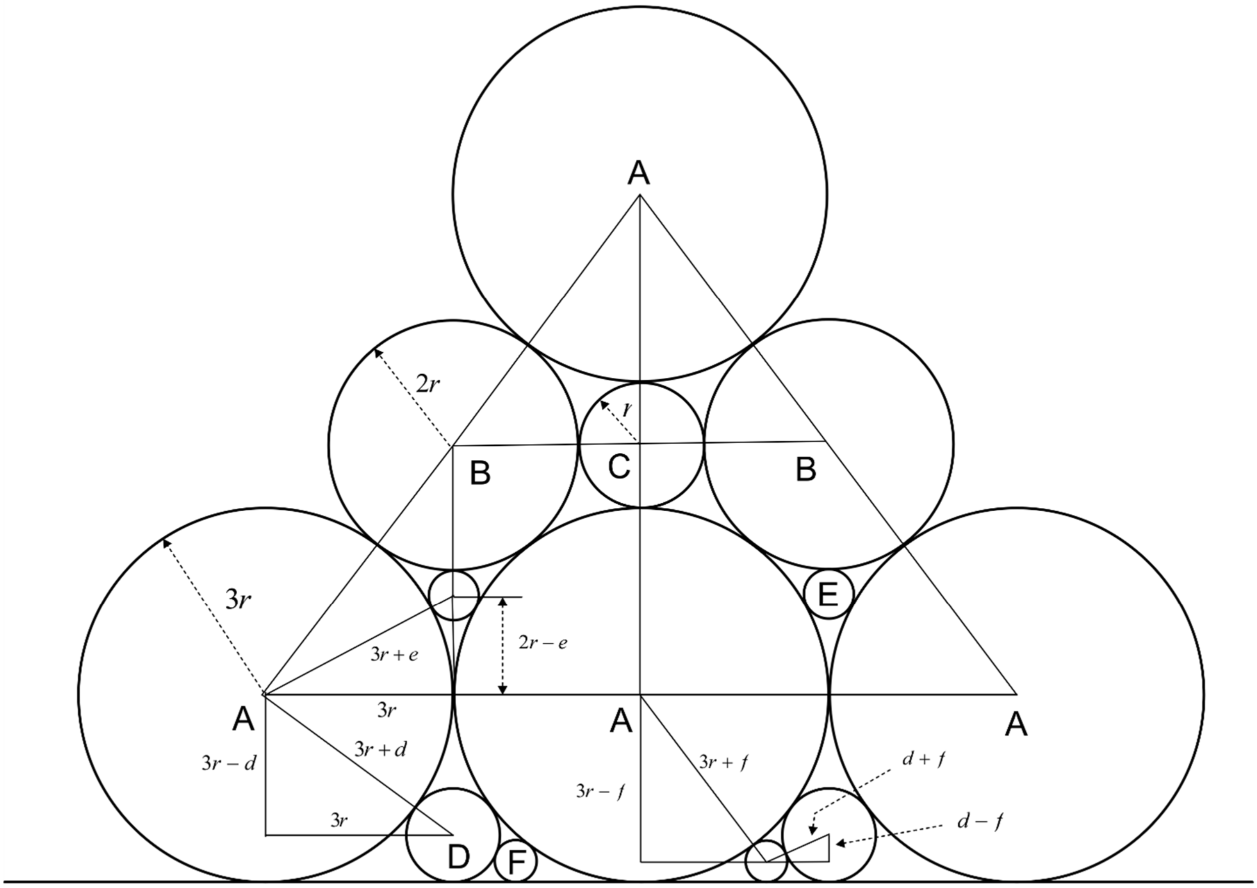


A factory makes six types of cylindrical pipe, A to F in decreasing size, whose diameters in centimetres are whole numbers, with type A 50 per cent wider than type B. The pipes are stacked in the yard as a touching row of As with an alternating row of touching Bs and Cs in the next layer, with each B touching two As. Type Ds fill the gap between the As and the ground; Es fill the gap between As and the Bs; and Fs fill the gap between As, Ds and the ground. Finally, another row of As is put on top of the stack, giving a height of less than 5 metres.

What is the final height of the stack in centimetres?

Solution by Brian Gladman



Since pipe A is 50% larger than pipe B, we can define their radii as $2r$ and $3r$ and we can see immediately from the geometry above that the radius of pipe C is r and the overall stack height is $14r$. Let the radii of the pipes D, E and F be d, e and f respectively.

Considering pipe D first, the illustrated right-angled triangle between the centres of left-hand pipes A and D gives:

$$(3r)^2 + (3r - d)^2 = (3r + d)^2$$

which simplifies to give $d = 3r/4$.

Now using the right-angled triangle on the diagonal between the centres of left-hand pipes A and E gives:

$$(3r)^2 + (2r - e)^2 = (3r + e)^2$$

which simplifies to $e = 2r/5$.

Finally considering the triangles between the centres of the lower middle A pipe, the right-hand F pipe and the following D pipe, we obtain two equations for the horizontal distances between the centres of A and F and the centres of F and D:

$$(h_{AF})^2 = (3r + f)^2 - (3r - f)^2 = 12rf$$

$$(h_{FD})^2 = (d + f)^2 - (d - f)^2 = 4df = 3rf$$

The sum of these two distances is $3r$, hence:

$$\sqrt{12rf} + \sqrt{3rf} = 3r$$

which, after squaring and simplifying, gives $f = r/3$.

Since the pipe ***diameters*** are integer, r must be a multiple of 30, giving the six pipe ***radii*** as:

$$90k, 60k, 30k, 45k/2\,, 12k \text{ and} 10k$$

with an overall stack height of $420k$ for some integer k.

Since the stack height is less than 5 meters, k must be 1, which gives a height of 420 cm as the solution. The pipe diameters are then $180, 120, 60, 45, 24$ and 20 cm.

Using Descartes Theorem and Soddy Circles

Consider any three mutually tangential circles p, q and r together with a fourth circle s mutually tangential to all of them. Descartes theorem can be used to show that:

$$s = \frac{pqr}{pq + qr + rp \pm 2\sqrt{pqr(p + q + r)}}$$

If the circle $r \to \infty$ (circle → straight line) this reduces to:

$$s = \frac{pq}{p + q \pm 2\sqrt{pq}}$$

Here the plus/minus on the square roots determines whether the fourth tangential circle is interior/exterior to the other three.

For the 3 circles $(A, A, line) \rightarrow (3r, 3r, \infty) \rightarrow d$:

$$d = \frac{9r^2}{6r + 2\sqrt{9r^2}} = \frac{3r}{4}$$

For the 3 circles $(A, A, B) \rightarrow (3r, 3r, 2r) \rightarrow e$:

$$e = \frac{18r^3}{21r^2 + 2\sqrt{144r^4}} = \frac{2r}{5}$$

For the 3 circles $(A, D, line) \rightarrow (3r, d, \infty) \rightarrow f$:

$$f = \frac{9r^2/4}{15r/4 + 2\sqrt{9r^2/4}} = \frac{r}{3}$$

With the same results as before.