Sunday Times Teaser 3116 - Poll Positions

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In an election for golf-club president, voters ranked all four candidates, with no voters agreeing on the rankings. Three election methods were considered.

Under First-past-the-post, since the first-preferences order was A, B, C, D, the president would have been A.

Under Alternative Vote, since A had no majority of first preferences, D was eliminated, with his 2nd and 3rd preferences becoming 1st or 2nd preferences for others. There was still no majority of 1st preferences, and B was eliminated, with his 2nd preferences becoming 1st preferences for others. C now had a majority of 1st preferences, and would have been president.

Under a Borda points system, candidates were given 4, 3, 2, or 1 points for each 1st, 2nd, 3rd or 4th preference respectively. D and C were equal on points, followed by B then A.

How many Borda points did each candidate receive?

Solution by John Crabtree

There are at most 24 voters since each voter gives a different order of preference for the four candidates. And there are at most six of these (ABCD, ABDC, ACBD, ACDB, ADBC, ADCB) in which A will be the first preference. Hence, with A, B, C and D as the numbers of first preference votes for the four candidates (with $A > B > C > D$), the maximum numbers of first preference votes for A, B, C and D are 6, 5, 4, and 3 respectively.

Since $B > C$ and C overtakes B when D is eliminated in the first round of AV voting, D must transfer at least 2 votes to C. Hence the minimum first preference votes for A, B, C and D are 5, 4, 3 and 2 respectively.

Hence the possible sets for (A, B, C, D) are (6, 5, 4, 3), (6, 5, 4, 2), (6, 4, 3, 2) and (5, 4, 3, 2). In order to have two transfers to C, the votes with D in first place must include DCAB and DCBA.

For (6, 5, 4, 3), there are 18 voters with a total of $180 = 4 \times 45$ Borda points. When D is eliminated in the first round of AV voting, two votes (DCAB and DCBA) transfer to C and D's third vote must transfer to A to ensure that C has more first place votes than B. A's six first preference votes give A 24 border points; the minimum Borda points for A from B's five (of a possible six) first preference votes is 2 x 1 + 2 x 2 + 1 x 3 = 9, while C's four first preference votes contribute a minimum of 2 x 2 + 2 x 1 = 6. Finally, D's three votes (DCAB, DCBA and DAxy) give a further 6. Hence A has a minimum of 45 Borda points, more than the average, which means that A cannot come last as required.

For (6, 5, 4, 2), there are 17 voters with $170 = 4 \times 42.5$ Borda points. Proceeding in the same way as above, A's minimum score in this case is $6 \times 4 + (2 \times 1 + 2 \times 2 + 1 \times 3) + (2 \times 1 + 2 \times 2) + (1 \times 1 + 1 \times 2) = 42$ Borda points which cannot put A in last place.

For (6, 4, 3, 2), there are 15 voters with $150 = 4 \times 37.5$ Borda points. In this case A's min score is $6 \times 4 + (2 \times 1 + 2 \times 2) + (2 \times 1 + 1 \times 2) + (1 \times 1 + 1 \times 2) = 37$ Borda points, which again cannot put A in last place.

For (5, 4, 3, 2) there are 14 voters with $140 = 4 \times 35$ Borda points. A's minimum score is now $5 \times 4 + (2 \times 1 + 2 \times 2) + (2 \times 1 + 1 \times 2) + (1 \times 1 + 1 \times 2) = 33$ Borda points. With $C = D$ and $D > B > A$, the only solution is $A = 33, B = 35$ and $C = D = 36$.

At this point the teaser is solved. As a check, one needs to find a set of votes.

The votes with D in first place are DCAB and DCBA.

Looking at A's Borda points requires BCDA, BDCA, BCAD, BDAC; CBDA, CDBA and CpAq.

Looking at C's Borda points requires ABDC, ADBC, ABCD, ADCB and ACrs.

Looking at B's Borda points requires CpAq = CDAB and ACrs = ACDB.

Hence this set of votes is unique.

Under AV, C has 8 first place votes and A has 6 first place votes.