Sunday Times Teaser 3091 – Birthday Money

by Howard Williams

Published Sunday December 19 2021

My two daughters were born on the same day but seven years apart. Every birthday, with only one year's exception, I have given them both five pounds for each year of their age. They are now grown-up, but I have continued to do this on their birthdays, except for the one year when I couldn't afford to give either of them anything. Averaged out over all of their birthdays, including the one for which they received nothing, my elder daughter has now received 21 per cent more per birthday than her younger sister.

How much in total have I given to my daughters as birthday presents?

Solution by Brian Gladman

Let the younger daughter's age be l and her age in the missed payment year be m. The total sums gifted to the two daughters are then:

$$5\left\{\frac{l(l\ +1)}{2}-m\right\}\ \text{ and }\ 5\left\{\frac{(l+7)(l+8)}{2}-(m+7)\right\} \tag{1}$$

The relationship between l and m can now be derived from the equation for the ratio of the total amounts gifted to the two daughters:

$$121\frac{l(l+1)-2m}{2l}=100\frac{(l+7)(l+8)-2(m+7)}{2(l+7)} \tag{2}$$

After simplification and rearrangement, this provides an equation for m in terms of l:

$$m=l\left\{\frac{(l+7)(21l-679)+1400}{14(3l+121)}\right\} \tag{3}$$

which, after further rearrangement, can be put in this form:

$$m=\frac{l}{6}\left\{\frac{(3l-38)^2-2881}{(3l+121)}\right\} \tag{4}$$

For m positive we must have $(3l-38)^2 \geq 2881$ which requires that $l \geq 31$.

For $l \geq 31$, m increases monotonically but we know that $m \leq l$ so we can find the maximum l value by setting $m = l$ in equation (4), which, after simplification, gives:

$$9l^2-246l-2163=(l+7)(3l-103)=0 \tag{5}$$

This gives $l_{max} = 103/3$, which means that $l \leq 34$. Hence the full range for integer l is $31 \leq l \leq 34$, leaving just four values to be checked.

The solution $l = 33$ and $m = 21$ is hence easily found, after which substitution into equation (1) gives the lifetime gifts for the two daughters as £2700 and £3960 with a total of £6660.