Sunday Times Brainteaser 1816 – Polls Apart

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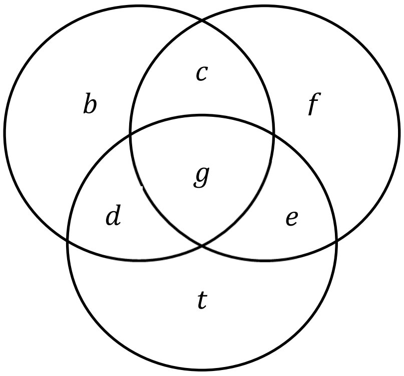
Published 6th July 1997

A TV station has commissioned a survey of a small sample of its viewers. One quarter said they watched boxing, one third said they watched tennis, and one half said they watched football. All watched at least one sport. When analysed, the results showed that the number of those questioned who watched just one of the sports equalled the square of the number watching all three. The TV people were unconvinced and asked for a second poll with the number of people questioned increased by 50%. Surprisingly, all that was said above about the first poll was still true for the second.

How many people were questioned in the first poll?

Solution by Brian Gladman



Let n be the total number of people in the first survey. The three circles in the Venn diagram to the left represent the total number of people who watch boxing, football and tennis respectively with b, f and t representing those who watch only these sports. The areas c, d and e represent those who watch two of the three sports while the central area g represents those who watch all three sports.

We can now set out equations for the statements made in the teaser:

$$b + c + d + g = n/4 \tag{1}$$

$$t + d + e + g = n/3 \tag{2}$$

$$f + c + e + g = n/2 \tag{3}$$

$$b + f + t = g^2 \tag{4}$$

$$n = (b + f + t) + (c + d + e) + g \tag{5}$$

We can now sum equations (1), (2) and (3) and rearrange the result to give:

$$(c + d + e) = 13\,n/24 - (b + t + f)/2 - 3\,g/2 \tag{6}$$

And using equations (4) and (6) to substitute into equation (5) we obtain:

$$11\,n/3 = 4g^2 - 4g = (2g - 1)^2 - 1 \tag{7}$$

In the second survey, the number of people is 50% larger $(\text{i.e. n}' = 3\,n/2)$ so we have a larger value of g, say g':

$$11n'/3 = 11\,n/2 = (2g' - 1)^2 - 1 \tag{8}$$

Eliminating n from equations (7) and (8) and re-arranging now gives:

$$(4g' - 2)^2 - 6(2g - 1)^2 = -2 \tag{9}$$

This is a generalised Pell equation which has an infinite sequence of solutions, the first of which are:

$$(4g' - 2), (2g - 1) = (2, 1), (22, 9), (218, 89), (2158, 881), \ldots \tag{10}$$

We need a solution that gives an integer n value and the first set of values to meet this requirement is $(218, 89)$, which gives $g = 45$ and $g' = 55$, finally giving $n = 2{,}160$ $(n' = 3{,}240)$.

The next solution is at $n = 211{,}680$ and $n' = 317{,}520$, which might even be considered a small sample for a national TV station!

While we know the sums $(b + f + t)$ and $(c + d + e)$, we cannot determine their sub-components.