Sunday Times Brainteaser 1746 – Party Time

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In the volatile world of Ruritanian politics, each of the three parties can safely expect to lose a fixed proportion of its supporters to each of the other parties from one election to the next. Thus, the Story party would retain a fixed proportion of its supporters, and lose fixed proportions (which may differ) to the Laboured party and to the Mauve Shirts, and so on. Three elections ago, the Story party and the Laboured party each won 40,000 votes and the Mauve Shirts won 20,000. Two elections ago, the Story party and the Mauve Shirts each won 35,000 votes, while the Laboured party won 30,000 votes. In the last election the Story party and the Laboured party each won 35,000 votes and the Mauve Shirts party won 30,000. The total electorate in the current election remains at 100,000.

What is the minimum number of votes the Mauve Shirts should expect to win in the current election?

Solution by Brian Gladman

Let the votes for the three parties be S, L and M and let p_{ab} be the percentage transferred from a to b at each election. The relationship between the current and next year (dashed) votes is then:

$$\begin{bmatrix} -(p_{sl}+p_{sm}) & p_{ls} & p_{ms} \\ p_{sl} & -(p_{ls}+p_{lm}) & p_{ml} \\ p_{sm} & p_{lm} & -(p_{ms}+p_{ml}) \end{bmatrix} \begin{bmatrix} S \\ L \\ M \end{bmatrix} = 100 \begin{bmatrix} S'-S \\ L'-L \\ M'-M \end{bmatrix} \qquad (1)$$

where the dashed votes are those for the next year. We can now establish the three equations for the vote three years ago, which after simplification are:

$$\begin{aligned} -2p_{sl} - 2p_{sm} + 2p_{ls} + p_{ms} &= -25 \\ 2p_{sl} - 2p_{ls} - 2p_{lm} + p_{ml} &= -50 \\ 2p_{sm} + 2p_{lm} - p_{ms} - p_{ml} &= 75 \end{aligned} \qquad (2)$$

Similarly, the equations for the vote two years ago are:

$$\begin{aligned} -7p_{sl} - 7p_{sm} + 6p_{ls} + 7p_{ms} &= 0 \\ 7p_{sl} - 6p_{ls} - 6p_{lm} + 7p_{ml} &= 100 \\ 7p_{sm} + 6p_{lm} - 7p_{ms} - 7p_{ml} &= -100 \end{aligned} \qquad (3)$$

With six variables and six equations, it seems at first that we can solve for the transfer percentages but two of the six equations are not independent so we have to solve in terms of two of the six percentages:

$$\begin{aligned} p_{ls} &= 7\,(p_{sl} + p_{sm} - 25)/8 \\ p_{lm} &= (625 - 7p_{sm})/8 \\ p_{ms} &= (p_{sl} + p_{sm} + 75)/4 \\ p_{ml} &= (250 - p_{sl})/4 \end{aligned} \qquad (4)$$

We now need the equations for the vote last year, which are:

$$\begin{aligned} S' &= 350(100 - p_{sl} - p_{sm} + p_{ls}) + 300p_{ms} \\ L' &= 350(100 + p_{sl} - p_{ls} - p_{lm}) + 300p_{ml} \\ M' &= 350(p_{sm} + p_{lm}) + 300(100 - p_{ms} - p_{ml}) \end{aligned} \qquad (5)$$

Substituting from equations (4) into (5) and simplifying now gives the current year vote predictions:

$$\begin{aligned} S' &= 125(1055 + p_{sl} + p_{sm})/4 \\ L' &= 125(1090 - p_{sl})/4 \\ M' &= 125(1055 - p_{sm})/4 \end{aligned} \qquad (6)$$

Multiplying the equations for L' and M' by 4 and considering the results modulo 4 shows that $p_{sl} \equiv 2 \bmod 4$ and $p_{sm} \equiv 3 \bmod 4$ which means that $p_{sl} = (4j+2)$ for any integer j in $0 \le j \le 24$ and $p_{sm} = (4k+3)$ for any integer k in $0 \le k \le 24$.

To find the minimum vote for M' we need the maximum possible value of p_{sm}. From equation (4) we can show that $4(p_{ms} + p_{ml}) = 325 + p_{sm}$, which shows that $p_{sm} \le 75$ (since $p_{ms} + p_{ml} \le 100$). Using this limit in the last equation in (6) hence shows that the minimum value of M' is 30625.