Sunday Times Teaser 3081 – Connect Four

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I have four different two-digit numbers, each having at least one digit which is a three. When I multiply any three of these numbers together I get a product that, with the inclusion of a leading zero, is one or more repetitions of the repetend of the reciprocal of the fourth two-digit number. A repetend is the repeating or recurring decimal of a number. For example, 1 divided by 27 is 0.037037……, giving a repetend of 037; in that case, the product would be 37 or 37037 or 37037037 etc.

What, in ascending order, are the four two-digit numbers?

Solution by Brian Gladman

Consider a two-digit integer d with a decimal reciprocal that consists of a recurring sequence of digits of length N starting immediately after the decimal point. If the recurring sequence as an integer is r, we then have:

$$\frac{1}{d} = r(10^{-n} + 10^{-2n} + \cdots) = \frac{10^{-n}r}{1 - 10^{-n}}$$

which simplifies to:

$$d \times r = 10^n - 1$$

From the teaser description we know that the integer r is the product of three two-digit numbers, which means that we are looking for an n value that allows $10^n - 1$ to be expressed as four two-digit numbers each of which contains at least one 3 digit. The minimum and maximum possible values are $13 \times 23 \times 30 \times 31 = 278{,}070$ and $63 \times 73 \times 83 \times 93 = 35{,}499{,}681$ so we know that $6 \leq n \leq 8$ and factoring $10^n - 1$ into its prime factors for these n values:

$$\begin{aligned} 10^6 - 1 &= \; 3^3 \times 7 \times 11 \times 13 \times 37 \\ 10^7 - 1 &= \; 3\text{^}2 \times 239 \times 4649 \\ 10^8 - 1 &= \; 3^2 \times 11 \times 73 \times 101 \times 137 \end{aligned}$$

shows that only $n = 6$ offers the possibility that $10^n - 1$ can be expressed as the product of four two-digit integers of the specified form.

For the $n = 6$ factors, the only way to 'absorb' the 7 to create a two-digit number with a three digit is $7 \times 9 = 63$ and the 11 can then be combined with the remaining three to give the four numbers and the answer as:

$$13 \times 33 \times 37 \times 63$$