Sunday Times Teaser 3702 - Dial M for Marriage

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George and Martha work in a town where phone numbers have seven digits. “That’s rare!” commented George. “If you look at your work number and mine, both exhibit only four digits of the possible ten (0-9 inclusive), each appearing at least once. Furthermore, the number of possible phone numbers with that property has just four single-digit prime number factors (each raised to a power where necessary) and those four numbers are the ones in our phone numbers.”

“And that is not all!” added Martha. “If you add up the digits in the two phone numbers you get a perfect number [1] in both cases. Both have their highest digits first, working their way down to the lowest.”

What are the two phone numbers?

[1] a perfect number equals the sum of its factors, e.g., 6 = 1 + 2 + 3]

Solution by Brian Gladman (with credit to Peter Noll for the combinatorial exercise).

Before we start, note that if we have n symbols that each occur $m_1, m_2, m_3, \ldots, m_n$ times, then the number of permutations of these symbols is given by:

$$\left(\sum_{i=1}^{n} m_i\right)! \Big/ \prod_{i=1}^{n} m_i!$$

Consider seven-symbol sequences that use only four different symbols (a, b, c, d), each at least once. There are three different unordered arrangements:

unordered symbols	number of ordered permutations	value (x 210)	multiplier (see below)	final multiplier
$aaaa\ b\ c\ d$	$7!/4!$	1	4	4
$aaa\ bb\ c\ d$	$7!/(3!\,2!)$	2	12	24
$aa\ bb\ cc\ d$	$7!/(2!)^3$	3	4	12

If we have four specific digits in the first arrangement, there will be four choices for a; allocations for b, c and d don’t add permutations since we have already counted their re-arrangements. Likewise for the second arrangement we have four choices for a and three choices for b giving a total 12 different choices. Finally, for the third arrangement we have four choices for d (again allocations to b, c and d rearrange the order of permutations but not their number). In total there are hence 40×210 ways of arranging seven digits using ***four specified digits***, each at least once.

There are now $10!/(4!\,6!)$ (= 210) ways of choosing any four digits from ten so the total number of ways of arranging seven digits using ***any four digits from ten***, each at least once, is $40 \times 210^2 = 1{,}764{,}000$.

We now factor 1,764,000 to find the digits in the telephone numbers and this gives $2^5 \times 3^2 \times 5^3 \times 7^2$, showing to that the two telephone numbers use the digits 2, 3, 5 and 7. Had we factored 8400 we would have obtained $2^3 \times 3 \times 5^2 \times 7$ giving the same four digits.

This confirms that the digits are the four single digit primes (as already specified in the teaser).

Since the minimum and maximum numbers using seven digits from 2, 3, 5 and 7 are 2222357 and 7777532, the minima and maxima for the digit sum are 23 and 38, which shows that the only digit sum that can be a perfect number is 28 (from 6, 28, 496, 8128, …).

From here it is straightforward to identify the only two sets of seven digits with this sum. And this in turn identifies the two phone numbers as 7753222 and 7553332 because they are in non-increasing digit order.

This is an intriguing teaser. The combinatorial exercise is somewhat ambiguous (1,764,000 or 8,400) but this doesn’t matter since either interpretation gives the same result. But it is redundant anyway since the answer it gives is provided in the teaser text!

I can’t help feeling disappointed that it is not necessary to solve this combinatorial exercise in order to solve the teaser. But maybe the author’s aim was to build an equivalent of the well-known ‘two trains’ puzzle (see https://mathworld.wolfram.com/TwoTrainsPuzzle.html). Like the teaser, this is one that provides two paths to the solution, one easy and one much more difficult!