Sunday Times Teaser 3071 - Three-Cornered Problem

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I have a set of equal-sized equilateral triangles, white on one face and black on the back. On the white side of each triangle I have written a number in each corner. Overall the numbers run from 1 to my age (which is less than thirty). If you picture any such triangle then it occurs exactly once in my set (for example, there is just one triangle containing the numbers 1, 1 and 2; but there are two triangles containing the numbers 1, 2 and 3).

The number of triangles that contain at least one even number is even.

The number of triangles that contain at least one odd number is odd.

The number of triangles that contain at least one multiple of four is a multiple of four.

How old am I?

Solution by Ciaran Lewis

If we label the triangle vertices with k different labels (A, B, C, …), there are three different types of triangle — k triangles with three identical labels (e.g. AAA), $k(k-1)$ triangles with two identical and one different label (e.g. AAB), and $k(k-1)(k-2)/3$ triangles with three different labels (e.g. ABC). We have to divide the last type total by 3 because each distinct triangle appears in the count three times in different rotations 120 degrees apart. Hence with k labels there are a total of $N(k) = k(k^2+2)/3$ triangles.

If the vertexes are labelled with the numbers $1..n$ with n even, there will be an equal number of even and odd labels and this will mean that the numbers of triangles with only even labels and only odd labels will be the same. This in turn will mean that the numbers of triangles with at least one even label and at least one odd label will also be the same. And, since one of these is even and the other is odd, we hence know that the number of labels must be odd with $(n-1)/2$ even and $(n+1)/2$ odd labels.

The number (T_E) of triangles with at least one even label is the difference between the total number of triangles and the number with only odd labels $N(n) - N(\{n+1\}/2) = (n-1)(7n^2+4n+9)/24$. In the same way, the number (T_O) of triangles with at least one odd label is $N(n) - N(\{n-1\}/2) = (n+1)\,(7n^2-4n+9)/24$.

If we define m and r such that $n = 4m + r$ with $r = 1$ or 3, we then have m numbers that are a multiple of 4 and $n - m$ that are not. Hence the number (T_{4M}) that have at least one multiple of 4 is given by $N(n) - N(n-m) = m\,(m^2 - 3mn + 3n^2 + 2)/3$.

Checking the parity of T_E and T_O as functions of n shows that T_E is even and T_O is odd only for n values in the sequence $1, 5, 9, 13, 17, \dots$ (i.e. $r = 1$ and $m = 1, 2, 3, 4, 5, \dots$). Inspection of T_{4M} as a function of m and n $(= 4m + 1)$ shows that it is a multiple of 4 only for the m values $4,\ 8, 12, \dots$ and n values $17, 33, 49, \dots$ where 17 is the only age less than 30. The corresponding T_O, T_E and T_{4M} values (which are not needed) are 1473, 1400 and 908 respectively. NB $T_O - T_E = (n^2+3)/4$ (n odd) and $T_O = T_E = n(7n^2+8)/24$ (n even).