Sunday Times Teaser 3064 June 13 2021, by Victor Bryant

The Turnip Prize is awarded to the best piece of work by an artist under fifty. This year's winning entry consisted of a mobile made up of many different plain white rectangular or square tiles hanging from the ceiling. The sides of the tiles were all whole numbers of centimetres up to and including the artist's age, and there was precisely one tile of each such possible size (where, for example, a 3-by-2 rectangle would be the same as a 2-by-3 rectangle). Last week one of the tiles fell and smashed and then yesterday another tile fell and smashed. However, the average area of the hanging tiles remained the same throughout.

How old is the artist?

All possible areas of 'green tiles' with sides a and b up till 5 cm:

a \ b	1	2	3	4	5
1	1	2	3	4	5
2		4	6	8	10
3			9	12	15
4				16	20
5					25
Area:	1	7	25	65	140
Number:	1	3	6	10	15

The cumulated area develops as: $1, 7, 25, 65, 140, \cdots$

This series is A001296, 4-dimensional pyramidal numbers. (Also obtainable by curve-fitting) as:

$$A(n) = \frac{1}{4}(3n+1)\binom{n+2}{3} = \frac{n(3n+1)(n+1)(n+2)}{24}$$

The number of tiles with sides from 1 to n cm is given by the sum of the triangular numbers:

$$S(n) = \frac{1}{2}n(n+1)$$

Thus, the average tile-area in terms of n, after reducing, is:

$$\overline{a}(n) = \frac{A(n)}{S(n)} = \frac{(3n+1)(n+2)}{12}$$

There are two ways that $\overline{a}(n)$ is a whole number:

Either by $$(n+2) \bmod 12 = 0 \;\Rightarrow\; n \in \{10, 22, 34, 46\}$$

or

$$(3n+1) \bmod 4 = 0 \;\wedge\; (n+2) \bmod 3 = 0$$

$$\Downarrow$$

$$n \in \{1, 5, 9, 13, \cdots\} \cap \{1, 4, 7, 13, \cdots\} \Rightarrow n \in \{13, 25, 37, 49\}$$

Of the resulting eight possible values for n, by inspection, only 37 cm (as longest possible side) also in its collection have two tiles with areas that equals the set's average:

$$\overline{a}(37) = 364 = 13 \times 28 = 14 \times 26 \quad (\text{cm}^2)$$

The artist is 37 years of age.

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