The Sunday Times Teaser 3054 – Discs A Go-Go

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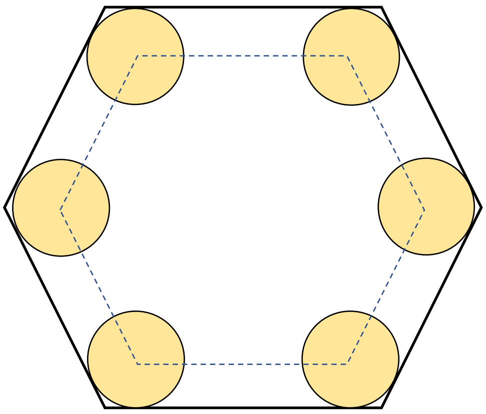
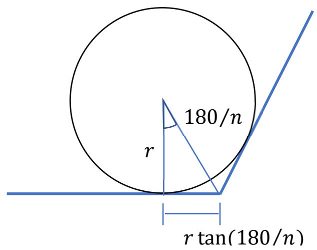
My kitchen floor is tiled with identically-sized equilateral triangle tiles while the floor of the bathroom is tiled with identically-sized regular hexagon tiles, the tiles being less than 1m across. In both cases the gaps between tiles are negligible. After much experimenting I found that a circular disc dropped at random onto either the kitchen or bathroom floor had exactly the same (non-zero) chance of landing on just one tile.

The length of each side of the triangular tiles and the length of each side of the hexagon tiles are both even triangular numbers of mm (ie, of the form 1+2+3+...).

What are the lengths of the sides of the triangular and hexagonal tiles?





Considering the hexagon as an example, the disc will lie entirely within the tile if its centre lies within the inner hexagon shown as a dotted line. We can see that this construction can be applied to any regular polyhedron.

The small lower diagram shows that if the side length of a regular polyhedron with n sides is s_n, its 'inner companion' will have a side length

$$s_n - 2r\tan(180/n)$$

Hence the probability p_n of falling entirely within any polyhedron is given by:

$$p_n = \left(\frac{s_n - 2r\tan(180/n)}{s_n}\right)^2 = \left(1 - \frac{2r\tan(180/n)}{s_n}\right)^2$$

If two polyhedra have the same probability we must then have:

$$\frac{\tan(180/m)}{s_m} = \frac{\tan(180/n)}{s_n}$$

This gives $s_3 = 3s_6$ for an equilateral triangle and a regular hexagon. If we let the sides of the triangle and the hexagon be triangular numbers t and h respectively we obtain the equation:

$$t(t+1) = 3h(h+1)$$

which, after considerable rearrangement, can be put in the form of a generalised Pell equation:

$$(2t+1)^2 - 3(2h+1)^2 = -2$$

The form of the solutions for such equations is well understood and if one solution can be found then an infinite sequence of solutions can be found using the following recurrence relations:

$$t_{n+1} = 2t_n + 3h_n + 2$$

$$h_{n+1} = t_n + 2h_n + 1$$

Since we can see that $(t, h) = (0,0)$ is a solution, we can generate the sequence of solutions:

$$(0,0), (3,1), (45,15), (630,210), (8778,2926), \ldots$$

Identifying even results less than 1000mm in size allows the solution to be identified as triangular tiles with sides of 630mm and hexagonal ones with sides of 210mm.