Sunday Times Teaser 3047 – Some Permutations

by Howard Williams

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I gave Robbie three different, single digit, positive whole numbers and asked him to add up all the different three-digit permutations he could make from them. As a check for him, I said that there should be three threes in his total. I then added two more digits to the number to make it five digits long, all being different, and asked Robbie's mother to add up all the possible five-digit permutations of these digits. Again, as a check, I told her that the total should include five sixes.

Given the above, the product of the five numbers was as small as possible.

What, in ascending order, are the five numbers?

Solution by Brian Gladman

First, we need to find the sum of the numbers formed by all the permutations of n given decimal digits. There are $n!$ such numbers which we can list as an addition sum with $n!$ rows and n columns. Each column consists of $n!$ digits with each of the n digits occurring an equal number of times, which means that all columns have the same sum of $(n-1)! \cdot \mathrm{sum}(digits)$. Noting that $(10^n - 1)/9$ is an n digit number consisting of n 1's (called a repunit), we can now evaluate the permutation sum as:

$$(n-1)! \ \cdot (10^n - 1)/9 \cdot \mathrm{sum}(digits) \tag{1}$$

Using this expression for the first choice of three positive digits, the sum of the numbers formed by their permutations is given by $222 \cdot \mathrm{sum}(digits)$ where:

$$\mathrm{sum}(1,2,3) \le \mathrm{sum}(digits) \le \mathrm{sum}(7,8,9) \tag{2}$$

and we are told that the result contains three 3's. But we can find[1] a better limit on the sum of the digits by noting that the permutation sum is an even four-digit number that contains three 3's; hence we have:

$$3330/222 \le \mathrm{sum}(digits) \le \ 3338/222 \tag{3}$$

This gives a digit sum of 15 which identifies eight possible three-digit numbers:

$$159 \ \ 168 \ \ 249 \ \ 258 \ \ 267 \ \ 348 \ \ 357 \ \ 456 \tag{4}$$

Now considering the final five positive digits, the sum of the numbers formed by their permutations is given by $266{,}664 \cdot \mathrm{sum}(digits)$ where we have:

$$\mathrm{sum}(1,2,3,4,5) \le \mathrm{sum}(digits) \le \mathrm{sum}(5,6,7,8,9) \tag{5}$$

We need a sum that includes five 6's and running through the possible sums identifies only one such value with a digit sum of 25. This shows that two (different) digits with a sum of 10 need to be added to our earlier three digits. This leads to four possibilities:

$$19 \ \ 28 \ \ 37 \ \ 46 \tag{6}$$

Since we need five different digits with the minimum possible product, we can simply note that the numbers in (4) and (6) are already listed in order of their digit products, which means we can scan both lists in increasing order to quickly find the pairs with different digits and a minimum overall digit product. This quickly yields two pairs with the same minimum digit product:

$$(1,5,9) + (2,8) \ \ ; \quad (2,5,8) + (1,9) \tag{7}$$

Both give the same solution for the teaser as $(1,2,5,8,9)$ with a product of 720.

[1] *credit: Erling Torkildsen.*