Sunday Times Teaser 3047 – Some Permutations
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Solution by Erling Torkildsen (15 February 2021)

The sum of numbers made of complete permutations of its *n* digits can be written as a product of a number and its digit-sum: $s_n = k_n(a_0 + a_1 + \cdots + a_{n-1})$ where *k* is given by:

$$k_n = \sum_{i=0}^{n-1} (n-1)! \cdot 10^i \quad \Rightarrow \quad \mathrm{k}_3 = 222 \text{ and } k_5 = 266{,}664$$

That way we have:

$$s_3 = 222 \cdot (\mathrm{a} + \mathrm{b} + \mathrm{c}) \quad \text{and} \quad s_5 = 266{,}664 \cdot (\mathrm{a} + \mathrm{b} + \mathrm{c} + \mathrm{d} + \mathrm{e})$$

As s_3 is even and contains three 3's we get:

$$\frac{3330}{222} \le a + b + c \le \frac{3338}{222} \quad \Rightarrow \quad a + b + c = 15$$

For s_5 we get $(a + b + c + d + e) = 15 + d + e \in \{18, \cdots, 32\}$

By inspection only $(a + b + c + d + e) = 25$ lets s_5 have five 6's $(266{,}664 \cdot 25 = 6{,}666{,}600)$

$(a + b + c) = 15 \Rightarrow (\mathrm{a,b,c}) \in \{(1,5,9), (1,6,8), (2,4,9), (2,5,8), (2,6,7), (3,4,8), (3,5,7), (4,5,6)\}$

$(d + e) = 25 - 15 = 10 \Rightarrow (\mathrm{d,e}) \in \{(1,9), (2,8), (3,7), (4,6)\}$

$\{a, b, c\}$ and $\{d, e\}$ have 16 disjoint unions where $\{1,5,9\} \cup \{2,8\}$ and $\{2,5,8\} \cup \{1,9\}$ give the smallest product of the members (720).

<u>The digits are 1, 2, 5, 8 and 9.</u>