Goldilocks and the Bear Commune

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In the bears' villa there are three floors, each with 14 rooms. The one switch in each room bizarrely toggles (on ↔ off) not only the single light in the room but also precisely two other lights on the same floor; moreover, whenever A toggles B, then B toggles A.

As Goldilocks moved from room to room testing various combinations of switches, she discovered that on each floor there were at least two separate circuits and no two circuits on a floor had the same number of lights. Furthermore, she found a combination of 30 switches that turned all 42 lights from 'off' to 'on', and on one floor she was able turn each light on by itself.

(a) How many circuits are there?
(b) How many lights are in the longest circuit?

Solution **Answer: (a) 7 (b) 10**

Justification: First observe that each circuit is a simple loop, like a necklace of lights. If there are n lights and switches in a circuit, label them as follows: Choose a random light as number 1, and choose 2 to be one of the lights toggled by switch 1. Since 2 toggles 1, choose 3 to be the other light toggled by 2, 4 the other light toggled by 3, and so on. Eventually we will reach the light labeled n, and this must toggle light 1 in order to satisfy the condition that each switch toggles exactly two other lights in addition to its own.

Also note that starting with all lights off, activating each switch once will turn all the lights on. This is because each light gets toggled three times.

We will now show that any combination of n lights (including single lights) can be switched on if and only if n is not a multiple of 3. Note that the number of light states, 2^n, is the same as the number of switch states. Therefore if there is a state cannot be reached, there must be two distinct sets of switches (starting from "all off") that produce the same light state. If these two sets are amalgamated, those in both (being applied twice) have no effect. The set remaining when those in both are removed - call it R - leaves the 'all off' state unchanged when its switches are activated, in other words, a choice of switches in which each light is toggled twice. Label the lights and switches from 1 to n so that. say, (switch/light) 2 is in R. Since light 2 is off, it must be toggled by just one adjacent switch, say by switch 3; thus 3 is in R and 1 is not in R. Since 3 is already toggled by itself and switch 2, it is not toggled by 4 and so 4 is not in R. Therefore 5 must toggle 4 and must be in R. Continuing this argument clockwise round the loop shows that R contains a sequence of neighbouring triples consisting of two in R followed by one not in R. The final sequence must eventually tie in with the light/switch 1 which is not in R. Hence the circuit is a amalgamation of these triples and so n is a multiple of three. Therefore if n is not a multiple of 3, all light states can be reached.

Now suppose that n is divisible by 3, say n = 3m. Since activating every third switch turns all the lights on from the "all off" state and since there are three ways of choosing every third switch, we certainly have two distinct switching sets with the same lighting outcome; then by the above argument there must be lighting states that are unobtainable. In particular single lights are unattainable because if they were, every state (which is a combination of suitable single lights) would be attainable. Note that m switches can turn "all off" to "all on" in this case, whereas n are required when n is not a multiple of 3 (by uniqueness and the fact that activating all switches do it).

The lighting circuits in the villa must be chosen from the following lengths (with minimal number switches to go from "all off" to "all on" shown in brackets):

14(14), 3&11(12), 4&10(14), 5&9(8), 6&8(10), 3&4&7(12), 3&5&6(8)

We can rule out 14 because there are at least 2 circuits on each floor. 4&10 is the only option with no circuits of length a multiple of 3 so that must be on the floor where just a single light can be switched on. Since 4&10 needs 14 switches to go from "all off" to "all on", the other two must need a total of 30 - 14 = 16 switches and by inspecting the numbers in brackets we obtain the unique solution

4&10, 5&9 and 3&5&6

on the three floors. There are 7 circuits, of which the longest is 10.