Sunday Times Teaser 3031 - End of the Beginning

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Jenny is using her calculator, which accepts the input of numbers of up to ten digits in length, to prepare her lesson plan on large numbers. She can't understand why the results being shown are smaller than she expected until she realizes that she has entered a number incorrectly.

She has entered the number with its first digit being incorrectly entered as its last digit. The number has been entered with its second digit first, its third digit second etc. and what should have been the first digit entered last. The number she actually entered into her calculator was 25/43rds of what it should have been.

What is the correct number?

Solution by John Crabtree

Let the first digit of the correct number (S) be f followed by the N digit number n. We hence have:

$$10n + f = \frac{25(10^N f + n)}{43} \tag{1}$$

This can be rearranged to give:

$$n = \frac{(25 \times 10^N - 43)f}{405} \tag{2}$$

We can now express the correct number S as:

$$S = f \times 10^N + \frac{(25 \times 10^N - 43)f}{405} = \frac{43(10^{N+1} - 1)f}{405} \tag{3}$$

Since S is an integer, both the numerator and denominator of this fraction must be divisible by 5 and this means that f must be 5 (the other factors cannot have 5 as a factor). Hence:

$$S = \frac{43(10^{N+1} - 1)}{81} \tag{4}$$

Numbers of the form $(10^k - 1)/9$ are known as repunits because they consist of sequences of k unit digits. Denoting a k digit repunit as R_k we can express S as:

$$S = \frac{43R_{N+1}}{9} \tag{5}$$

For a repunit to be divisible by 3, its number of digits must also be divisible by 3, which means that R_{N+1} is one of 111, 111111, 111111111 … The first of these numbers that is divisible by 9 is 111,111,111 which gives:

$$S = 530{,}864{,}197 \tag{6}$$