Please Mind the Gap

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Ann, Beth and Chad start running clockwise around a 400m running track. They run at a constant speed, starting at the same time and from the same point; ignore any extra distance run during overtaking.

Ann is the slowest, running at a whole number speed below 10 m/s, with Beth running exactly 42% faster than Ann, and Chad running the fastest at an exact percentage faster than Ann (but less than twice her speed).

After 4625 seconds, one runner is 85m clockwise around the track from another runner, who is in turn 85m clockwise around the track from the third runner.

They decide to continue running until gaps of 90m separate them, irrespective of which order they are then in.

For how long in total do they run (in seconds)?

Solution **Answer: 7250 seconds**

Let A’s speed be “a” and for simplification change the frame of reference by deducting A’s speed from each of the three runners’ speeds so that A remains at the starting position – point 0.
B’s speed is now 1.42a – a = 0.42a. If after 4625s of running distances are 85m apart then
4625 x 0.42a/400 or 1942.5a/400 must have a remainder of 85, 170, 230 or 315.
From inspection “a” must be an even number and the following table shows the remainders for even single digit values of “a” –

	Value of "a"			
	2	4	6	8
Distance run - 1942.5a	3885	7770	11655	15540
Remainder, distance run divided by 400	285	**170**	55	340

It can be seen that “a” must be 4m/s as only this speed gives a valid remainder value.

The position of A after 4625s is 0, that of B is 170 so C’s position must be 85.
Let C’s excess speed relative to A be “c”, where $0.42<c<1$ since C’s speed is less than twice that of A.
Therefore c x 4 x 4625/400 must have a remainder of 85.
This converts to 18500c = 400k + 85 which becomes 3700c – 80k = 17 for some whole number k.
Now let C = 100c, where $42<C<100$, then 37C-80k = 17
37C must end in 7, so C ends in 1 and C = 51, 61, 71, 81 or 91.
The only value that gives a whole number for k is C=61.

Let the total running time for the runners to be 90m apart be “t”.
At time t A’s position is 0, B’s position is the remainder from 0.42x4xt/400 and C’s position the remainder from 0.61x4xt/400.
The allowable remainder values are 90, 180, 220 and 310, so
For B, 1.68t = 400y + 90, 180, 220 or 310, for some whole number y, and correspondingly
For C, 2.44t = 400z + 180|310, 90, 310 or 90|220 (the order of the runners, clockwise, being ABC|CAB, ACB, BCA or BAC|CBA) for some whole number z

To eliminate “t” in the above, multiply the first equation by 61 and the second by 42 to get
102.48t = 24400y + 61(90, 180, 220 or 310) = 16800z + 42(180|310, 90, 310 or 90|220)

The first and fourth possibilities don’t work, as 61x90 and 61x310 aren’t divisible by 20 and all other terms are. We are left with:

2440y + (1098 or 1342) = 1680z + (378 or 1302), which reduce to:
(1) 61y + 18 = 42z
(2) 61y + 1 = 42z

The first solution of (1) is y=30, z=44 and t = (400y+180)/1.68 = 7250
The first solution of (2) is y=11, z=16 and t = (400y+220)/1.68 = 2750
The next solution of (2) is y=53, z=77 and t = (400y+220)/1.68 = 12750

Therefore the first time after 4625s that there will be gaps of 90m is at a time of 7250s